A GENERAL METHOD OF SOLVING THE PLANE ELASTO-PLASTIC PROBLEM*)

by

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1. *Introduction*

The plane elasto-plastic problem is a simplified formulation of the problem to determine the stresses in some medium around an infinite cylindrical hole. Assuming a state of plane strain the problem is reduced to a formulation in the plane of the complex variable $z=x+iy$ with a hole bounded by some closed curve B. It is required to find the stresses taking assigned values on the boundary B and at infinity, which conditions give rise to a plastic region that surrounds the hole and an elastic region being the remaining exterior. The main problem is to determine the contour C that separates the plastic from the elastic region such that the stresses are continuous functions throughout the exterior of the boundary B. If this contour has been found the problem divides into a pure plastic problem and a pure elastic one so that consequently the stresses can easily be obtained.

In the following we are concerned with a general method of solving the plane elasto-plastic problem. After having dealt with the basic equations of elasticity and plasticity the relations on which the method is based are derived. Finally we will be occupied with some indications concerning the numerical treatment of these relations.

The stresses are described by means of the elements σ_x, σ_y and τ_{xy} of a symmetric stress-tensor. These elements satisfy the equilibrium-conditions:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{1}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{2}$$

which equations are fulfilled by the Airy stress-function $U(x, y)$ if

$$\sigma_x = U_{yy}$$

$$\sigma_y = U_{xx}$$

$$\tau_{xy} = -U_{xy}$$

In elasticity the function U is a biharmonic function, whereas in plasticity this function satisfies a hyperbolic differential equation.

If φ denotes the angle of rotation between the coordinate-axes and the two orthogonal directions of the principal stresses σ_1 and σ_2 at some point (x, y) then

$$\sigma_x = \sigma + \rho \cos 2\varphi \tag{3}$$

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$$\sigma_y = \sigma - \rho \cos 2\varphi \tag{4}$$

$$\tau_{xy} = \rho \sin 2\varphi \tag{5}$$

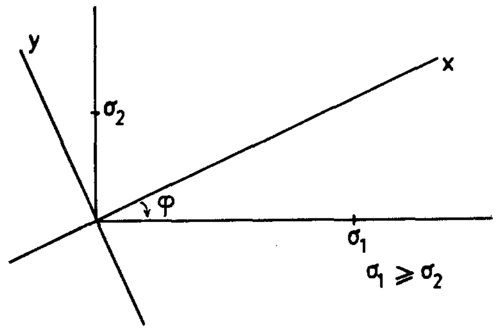

Fig. 1.

where for brevity we have put

$$\sigma = \tfrac{1}{2}(\sigma_1 + \sigma_2) \tag{6}$$

$$\rho = \tfrac{1}{2}(\sigma_1 - \sigma_2) \tag{7}$$

In addition to the equations (1) and (2) Hooke's law must be fulfilled in the elastic region which leads to the biharmonic equation for the stress-function U. In the plastic region however a plasticity-condition must be satisfied. This means that ρ (the radius of Mohr's circle) is some given function of σ from which two hyperbolic equations for σ and φ will be obtained.

2. *The basic equations of elasticity*

We shall briefly go into the connection between the plane elastic problem and the theory of functions of a complex variable. As indicated in the preceding section the stress-function U satisfies the biharmonic equation

$$\Delta \Delta U = 0 \tag{8}$$

Therefore two functions φ and χ exist such that

$$U = \mathrm{Re}\left[\bar{z}\varphi(z) + \chi(z)\right] \tag{9}$$

where $\bar{z}$ denotes the conjugate of the complex variable z. From this result the following relations of Kolosov-Muskhelishvili are obtained:

$$\sigma_x + \sigma_y = 2\left[\varphi'(z) + \overline{\varphi'(z)}\right] \tag{10}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\bar{z}\varphi''(z) + \chi''(z)\right] \tag{11}$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2\left[z\overline{\varphi''(z)} + \overline{\chi''(z)}\right] \tag{12}$$

In plane elasticity one of the fundamental problems is the determination of the elastic equilibrium, given the resultant vector (X, Y) of the external stresses applied to the boundary of some cavity and the stresses at infinity. If ψ denotes the derivative χ' then the functions φ and ψ must have the form

$$\varphi(z) = -\frac{X + iY}{2\pi(1+\kappa)} \log z + \left[\frac{p+q}{4} + ic\right] z + \varphi_o(z) \tag{13}$$

$$\psi(z) = \frac{\kappa(X - iY)}{2\pi(1+\kappa)} \log z + \left[-\frac{p-q}{2} + it\right] z + \psi_o(z) \tag{14}$$

Here φ_0 and ψ_0 are functions holomorphic in the exterior of the cavity (it is assumed that the origin of coordinates lies in the interior). The constants p, q and t are the stresses at infinity:

$$\sigma_x(\infty) = p$$

$$\sigma_y(\infty) = q$$

$$\tau_{xy}(\infty) = t$$

and c is an unessential constant that does not influence the stresses. Because we have to do with the derivatives of the functions φ and ψ we shall denote these derivatives by φ and ψ instead of the functions itself. Thus the equations of Kolosov-Muskhelishvili become

$$\sigma_x + \sigma_y = 2\left[\varphi(z) + \overline{\varphi(z)}\right] \tag{15}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\bar{z}\varphi'(z) + \psi(z)\right] \tag{16}$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2\left[z\overline{\varphi'(z)} + \overline{\psi(z)}\right] \tag{17}$$

where φ and ψ have the form

$$\varphi(z) = -\frac{X+iY}{2\pi(1+\kappa)}\, 1/z + \left[\frac{p+q}{4} + ic\right] + \varphi_0(z) \tag{18}$$

$$\psi(z) = \frac{\kappa(X-iY)}{2\pi(1+\kappa)}\, 1/z + \left[-\frac{p-q}{2} + it\right] + \psi_0(z) \tag{19}$$

the functions φ_0 and ψ_0 being represented by

$$\varphi_0(z) = \sum_{k=2}^{\infty} \frac{\alpha_k}{z^k} \tag{20}$$

$$\psi_0(z) = \sum_{k=2}^{\infty} \frac{\beta_k}{z^k} \tag{21}$$

3. The basic equations of plasticity

As pointed out in the first section the plastic region in the exterior of the boundary B of the hole is that region where the plasticity-condition is satisfied. This condition defines ρ as a function of σ from which we shall derive two hyperbolic differential equations for the quantities σ and φ.

According to (3) and (5) the stresses σ_x and τ_{xy} are respectively

$$\sigma + \rho \cos 2\varphi$$

and

$$\rho \sin 2\varphi$$

Differentiating with respect to x and y we obtain from (1) if $\rho' = \frac{d\rho}{d\sigma}$

$$(1 + \rho' \cos 2\varphi)\frac{\partial\sigma}{\partial x} - 2\rho \sin 2\varphi \frac{\partial\varphi}{\partial x} + \rho' \sin 2\varphi \frac{\partial\sigma}{\partial y} + 2\rho\cos 2\varphi \frac{\partial\varphi}{\partial y} = 0 \tag{22}$$

The stresses τ_{xy} and σ_y are respectively

$$\rho \sin 2\varphi$$

and

$$\sigma - \rho \cos 2\varphi$$

Again differentiating with respect to x and y it follows from (2) that

$$\rho' \sin 2\varphi \frac{\partial\sigma}{\partial x} + 2\rho\cos 2\varphi \frac{\partial\varphi}{\partial x} + (1-\rho'\cos 2\varphi)\frac{\partial\sigma}{\partial y} + 2\rho\sin 2\varphi\frac{\partial\varphi}{\partial y} = 0 \qquad (23)$$

The equations (22) and (23) constitute two quasi-linear equations of the first order. In order to find the characteristics we utilize the property that in a linear combination of (22) and (23) differentiation occurs in a characteristic direction. Multiplying the equation (22) by sin λ and the equation (23) by cos λ and subtracting them we obtain, leaving the significance of λ as it is

$$\begin{aligned}&\left[-\sin\lambda + \rho'\sin(2\varphi-\lambda)\right]\frac{\partial\sigma}{\partial x} + 2\rho\cos(2\varphi-\lambda)\frac{\partial\varphi}{\partial x} + \\ &\left[\cos\lambda - \rho'\cos(2\varphi-\lambda)\right]\frac{\partial\sigma}{\partial y} + 2\rho\sin(2\varphi-\lambda)\frac{\partial\varphi}{\partial y} = 0 \qquad (24)\end{aligned}$$

Setting

$$\frac{\cos\lambda - \rho'\cos(2\varphi-\lambda)}{-\sin\lambda + \rho'\sin(2\varphi-\lambda)} = \frac{\sin(2\varphi-\lambda)}{\cos(2\varphi-\lambda)} = \operatorname{tg}\mu$$

we obtain

$$-\sin\lambda\sin(2\varphi-\lambda) + \rho'\sin^2(2\varphi-\lambda) = \cos\lambda\cos(2\varphi-\lambda) - \rho'\cos^2(2\varphi-\lambda)$$

or

$$\cos 2(\varphi-\lambda) = \rho' \qquad (25)$$

Thus for real characteristics the condition $|\rho'| < 1$ must be satisfied. However this is obvious from the fact that

$$\frac{d\rho}{d\sigma} = \cos\alpha \qquad (26)$$

where α is the complement of the angle between the tangent at the envelope of the circles with radius $\rho(\sigma)$ and the σ-axis in Mohr's diagram.
Hence

$$\cos 2(\varphi-\lambda) = \cos\alpha$$

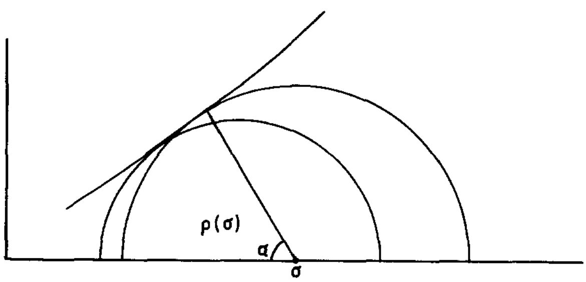

Fig. 2.

and consequently

$$2(\varphi-\lambda) = \pm\alpha \qquad (27)$$

from which we find that

$$\operatorname{tg} \mu = \operatorname{tg} (2\varphi - \lambda) = \operatorname{tg} (\varphi \mp \frac{\alpha}{2})$$

so that

$$\mu = \varphi \pm \frac{\alpha}{2} \tag{28}$$

Thus we obtain from the equation (24):

$$\left[-\sin(\varphi \pm \frac{\alpha}{2}) + \cos\alpha \sin(\varphi \mp \frac{\alpha}{2})\right] \frac{\partial\sigma}{\partial x} + 2\rho \cos(\varphi \mp \frac{\alpha}{2})\frac{\partial\varphi}{\partial x} +$$

$$\left[\cos(\varphi \pm \frac{\alpha}{2}) - \cos\alpha \cos(\varphi \mp \frac{\alpha}{2})\right] \frac{\partial\sigma}{\partial y} + 2\rho \sin(\varphi \mp \frac{\alpha}{2})\frac{\partial\varphi}{\partial y} = 0$$

Applying simple goniometric substitutions the following formulas for the characteristic equations

$$\sin\alpha \left[\cos\mu_1 \frac{\partial\sigma}{\partial x} + \sin\mu_1 \frac{\partial\sigma}{\partial y}\right] + 2\rho \left[\cos\mu_1 \frac{\partial\varphi}{\partial x} + \sin\mu_1 \frac{\partial\varphi}{\partial y}\right] = 0$$

and

$$-\sin\alpha \left[\cos\mu_2 \frac{\partial\sigma}{\partial x} + \sin\mu_2 \frac{\partial\sigma}{\partial y}\right] + 2\rho \left[\cos\mu_2 \frac{\partial\varphi}{\partial x} + \sin\mu_2 \frac{\partial\varphi}{\partial y}\right] = 0,$$

where $\mu_1 = \varphi + \frac{\alpha}{2}$ and $\mu_2 = \varphi - \frac{\alpha}{2}$, are easily derived. These equations can also be written in the form

$$\sin\alpha \frac{\partial\sigma}{\partial\mu_1} + 2\rho \frac{\partial\varphi}{\partial\mu_1} = 0 \tag{29}$$

$$-\sin\alpha \frac{\partial\sigma}{\partial\mu_2} + 2\rho \frac{\partial\varphi}{\partial\mu_2} = 0 \tag{30}$$

Putting finally

$$\Phi(\sigma) = \int \frac{\sin\alpha}{2\rho} d\sigma \tag{31}$$

the characteristic equations take the simple form:

$$\frac{\partial\Phi}{\partial\mu_1} + \frac{\partial\varphi}{\partial\mu_1} = 0 \tag{32}$$

$$\frac{\partial\Phi}{\partial\mu_2} - \frac{\partial\varphi}{\partial\mu_2} = 0 \tag{33}$$

Thus the stresses, taking assigned values on the boundary B, can easily be found by means of the equations (29) and (30) or (32) and (33) in the plastic region.

4. *The relations on which the method is based*

In this section we derive the basic relations for the method to deter-

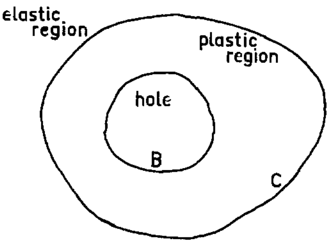

Fig.3.

mine the unknown contour C that separates the plastic from the elastic region. The equations of Kolosov-Muskhelishvili:

$$\sigma_x + \sigma_y = 2\left[\varphi(z) + \overline{\varphi(z)}\right] \tag{34}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\bar{z}\,\varphi'(z) + \psi(z)\right] \tag{35}$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2\left[z\,\overline{\varphi'(z)} + \overline{\psi(z)}\right] \tag{36}$$

hold in the elastic region, especially on the contour C. Thus, if we consider this contour, the stresses in the left members can be taken as being determined from the assigned values on the boundary B by the method of characteristics in the plastic region, whereas the functions φ and ψ refer to the elastic region. In this sense the equations (34), (35) and (36) are only valid on the contour C.

If $z=\omega(\zeta)$ is a conformal mapping that transforms the (unknown) contour C in the complex z-plane into the unit-circle in the complex ζ-plane then the Fourier-series of both members of the equations (34), (35) and (36) must be identical on the unit-circle in the ζ-plane. As regards this conformal mapping it is assumed that the exterior of the contour C in the z-plane is mapped onto the interior of the unit circle in the complex ζ-plane such that $\omega(o)=\infty$. Because the mapping is a one-to-one correspondence having a simple pole at $\zeta=o$ the function $\omega(\zeta)$ can be represented by:

$$\omega(\zeta) = \frac{c}{\zeta} + \gamma_o + \gamma_1\,\zeta + \gamma_2\,\zeta^2 + \gamma_3\zeta^3 + \ldots \tag{37}$$

Since the argument of the coefficient c is arbitrary this constant can be assumed to be positive:

$$c > o$$

Thus the point is to determine a conformal mapping such that both members of the equations (34), (35) and (36) are identical on the unit-circle in the ζ-plane. Indeed the equations of Kolosov-Muskhelishvili are then fulfilled on the contour C in the z-plane. Suppose the left members of the equations (34), (35) and (36) to be mapped on the Fourier-series:

$$\sigma_x + \sigma_y \longrightarrow F(\sigma) = \sum_{-\infty}^{\infty} c_n\,\sigma^n,\ (c_{-n} = \bar{c}_n) \tag{38}$$

and

$$\sigma_y - \sigma_x - 2i\tau_{xy} \longrightarrow G(\sigma) = \sum_{-\infty}^{\infty} d_n\,\sigma^n \tag{39}$$

so that

$$\sigma_y - \sigma_x + 2i\tau_{xy} \longrightarrow \overline{G(\sigma)} = \sum_{-\infty}^{\infty} \bar{d}_{-n}\sigma^n \tag{40}$$

where $\sigma = e^{i\theta}$. The functions φ and ψ in the right members of the equations (34), (35) and (36) are power-series of $1/z$. Since

$$\frac{1}{z^k} = \frac{1}{\left[\frac{c}{\zeta} + \gamma_0 + \gamma_1\zeta + \gamma_2\zeta^2 + \ldots\right]^k} = \frac{\zeta^k}{\left[c + \gamma_0\zeta + \gamma_1\zeta^2 + \gamma_2\zeta^3 + \ldots\right]^k}$$

where $\left[c+\gamma_0\zeta+\gamma_1\zeta^2+\gamma_2\zeta^3+\ldots\right]^{-1}$ can be written as a series of non-negative powers of ζ it follows that φ and ψ have the expansions

$$\varphi(\zeta) = \sum_{k=0}^{\infty} a_k\zeta^k \tag{41}$$

and

$$\psi(\zeta) = \sum_{k=0}^{\infty} b_k\zeta^k \tag{42}$$

where obviously $\mathrm{Re}(a_o) = \frac{p+q}{4}$, $b_o = -\frac{1}{2}(p-q)+it$ and $b_1 = -\kappa\bar{a}_1$. Thus it follows from the equation (34) that

$$2a_n = c_n, \quad n \geqslant 1 \tag{43}$$

Moreover

$$c_o = p + q \tag{44}$$

and

$$b_1 = -\kappa\bar{a}_1 \tag{45}$$

If the resultant vector (X, Y) of the external stresses at the boundary B vanishes then apparently the coefficients a_1 and b_1 vanish. Since

$$\varphi'(\zeta) = \frac{d\varphi}{d\zeta} = \frac{d\varphi}{dz} \cdot \frac{dz}{d\zeta} = \varphi'(z) \cdot \omega'(\zeta)$$

the equations (34), (35) and (36) become after the conformal transformation

$$F(\sigma) = 2\left[\varphi(\sigma) + \overline{\varphi(\sigma)}\right] \tag{46}$$

$$G(\sigma) == 2\left[\frac{\omega}{\overline{\omega}'}\,\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)}\right] \tag{47}$$

$$\overline{G(\sigma)} = 2\left[\frac{\overline{\omega}}{\omega'}\,\varphi'(\sigma) + \psi(\sigma)\right] \tag{48}$$

Now applying the Cauchy-operator to the equation (47):

$$\frac{1}{2\pi i}\oint \frac{G(\sigma)d\sigma}{\sigma - \zeta} = \frac{1}{\pi i}\oint \frac{\omega}{\overline{\omega}'}\,\frac{\overline{\varphi'(\sigma)}}{\sigma - \zeta}\,d\sigma + \frac{1}{\pi i}\oint \frac{\overline{\psi(\sigma)}d\sigma}{\sigma - \zeta}$$

where $|\zeta| < 1$ we find that

$$\frac{1}{2\pi i}\oint \frac{G(\sigma)d\sigma}{\sigma - \zeta} = \frac{1}{2\pi i}\oint G(\sigma)\sum_{k=0}^{\infty}\frac{\zeta^k}{\sigma^{k+1}}\,d\sigma = \frac{1}{2\pi}\int_0^{2\pi}\sum_{l=-\infty}^{\infty}\sum_{k=0}^{\infty} d_l\sigma^l\frac{\zeta^k}{\sigma^{k+1}}\,\sigma d\theta =$$
$$= \sum_{n=0}^{\infty} d_n\zeta^n$$

and

$$\frac{1}{2\pi i}\oint\frac{\overline{\psi(\sigma)}d\sigma}{\sigma-\zeta}=\frac{1}{2\pi i}\oint\overline{\psi(\sigma)}\sum_{k=0}^{\infty}\frac{\zeta^k}{\sigma^{k+1}}d\sigma=\frac{1}{2\pi}\int_0^{2\pi}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\bar{b}_l}{\sigma^l}\frac{\zeta^k}{\sigma^{k+1}}\sigma d\theta=$$
$$\bar{b}_0=-\frac{p-q}{2}-it$$

Putting

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}=\sum_{-\infty}^{\infty}\lambda_n\sigma^n$$

and passing to the limit $|\zeta|=1$ we find, since

$$\frac{1}{2\pi i}\oint\frac{\omega}{\overline{\omega'}}\frac{\overline{\varphi'(\sigma)}}{\sigma-\zeta}d\sigma=\frac{1}{2\pi i}\oint\frac{\omega}{\overline{\omega'}}\overline{\varphi'(\sigma)}\sum_{k=0}^{\infty}\frac{\zeta^k}{\sigma^{k+1}}d\sigma=\frac{1}{2\pi}\int_0^{2\pi}\sum_{l=-\infty}^{\infty}\sum_{k=0}^{\infty}\lambda_l\sigma^l\frac{\zeta^k}{\sigma^{k+1}}\sigma d\theta=$$
$$=\sum_{n=0}^{\infty}\lambda_n\zeta^n$$

that

$$2\lambda_n=d_n,\quad n\geqslant 1 \tag{49}$$
$$2\lambda_0=d_0+p-q+2it \tag{50}$$

In the same way we find from (48) that

$$\bar{d}_{-n}=2\bar{\lambda}_{-n}+2b_n,\quad n\geqslant 1.$$

Thus we have from (45) especially:

$$2\lambda_{-1}=d_{-1}+2\ \kappa a_1 \tag{51}$$

We have put

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}=\sum_{-\infty}^{\infty}\lambda_n\sigma^n \tag{52}$$

Therefore:

$$\frac{\left[\frac{c}{\sigma}+\gamma_0+\gamma_1\sigma+\gamma_2\sigma^2+\dots\right]}{\left[-c\sigma^2+\bar{\gamma}_1+\frac{2\bar{\gamma}_2}{\sigma}+\frac{3\bar{\gamma}_3}{\sigma^2}+\dots\right]}\left[\bar{a}_1+\frac{2\bar{a}_2}{\sigma}+\frac{3\bar{a}_3}{\sigma^2}+\dots\right]=\sum_{-\infty}^{\infty}\lambda_n\sigma^n$$

or

$$\left[\frac{c}{\sigma}+\gamma_0+\gamma_1\sigma+\gamma_2\sigma^2+\dots\right]\left[\bar{a}_1+\frac{2\bar{a}_2}{\sigma}+\frac{3\bar{a}_3}{\sigma^2}+\dots\right]=$$
$$\left[-c\sigma^2+\bar{\gamma}_1+\frac{2\bar{\gamma}_2}{\sigma}+\frac{3\bar{\gamma}_3}{\sigma^2}+\dots\right]\sum_{-\infty}^{\infty}\lambda_n\sigma^n$$

Equating the coefficients of σ^k in both members we find, if $k\geqslant 0$:

$$\sum_{m=1}^{\infty}m\bar{a}_m\gamma_{m+k-1}=-\lambda_{k-2}c+\sum_{m=1}^{\infty}m\lambda_{m+k-1}\bar{\gamma}_m \tag{53}$$

From (49), (50) and (51) it is evident that

$$k>0 \tag{54}$$

Thus neglecting terms of order higher than n in the expansion of $\omega(\zeta)$ we obtain the relations:

$$\left.\begin{array}{lll}
k=1: & \bar{a}_1\gamma_1 + 2\bar{a}_2\gamma_2 + 3\bar{a}_3\gamma_3 + \ldots + n\bar{a}_n\gamma_n & = -\lambda_{-1}c + \lambda_1\bar{\gamma}_1 + 2\lambda_2\bar{\gamma}_2 + \ldots + n\lambda_n\bar{\gamma}_n \\
k=2: & \bar{a}_1\gamma_2 + 2\bar{a}_2\gamma_3 + \ldots + (n-1)\bar{a}_{n-1}\gamma_n & = -\lambda_o c + \lambda_2\bar{\gamma}_1 + 2\lambda_3\bar{\gamma}_2 + \ldots + n\lambda_{n+1}\bar{\gamma}_n \\
\text{etc.} & - - - - - - - & \\
 & - - - - - - & \\
 & & \\
 & \bar{a}_1\gamma_n & = -\lambda_{n-2}c + \lambda_n\bar{\gamma}_1 + 2\lambda_{n+1}\bar{\gamma}_2 + \ldots + n\lambda_{2n-1}\bar{\gamma}_n
\end{array}\right\} \quad (55)$$

If we consider the constants a_k and λ_k as coefficients then we have a set of n homogeneous equations. Evidently γ_o does not occur in these equations. This is explained by the fact that

$$\frac{\overline{\varphi'(\zeta)}}{\overline{\omega'(\zeta)}} = \frac{\bar{a}_1\bar{\zeta} + 2\bar{a}_2\bar{\zeta}^2 + 3\bar{a}_3\bar{\zeta}^3 + \ldots}{-\frac{c}{\bar{\zeta}^2} + \bar{\gamma}_1 + 2\bar{\gamma}_2\bar{\zeta} + 3\bar{\gamma}_3\bar{\zeta}^2 + \ldots} = \frac{\bar{\zeta}^2\left[\bar{a}_1\bar{\zeta} + 2\bar{a}_2\bar{\zeta}^2 + 3\bar{a}_3\bar{\zeta}^3 + \ldots\right]}{-c + \bar{\gamma}_1\bar{\zeta}^2 + 2\bar{\gamma}_2\bar{\zeta}^3 + 3\bar{\gamma}_3\bar{\zeta}^4 + \ldots} = \sum_{n=3}^{\infty} \mu_n \bar{\zeta}^n$$

is a power-series of $\bar{\zeta}$ in which the constant term μ_o vanishes. Indeed the value of γ_o does not affect the integral

$$\frac{1}{2\pi i}\oint \frac{\gamma_o}{\bar{\omega}'} \frac{\bar{\varphi}'}{\sigma - \zeta}\, d\sigma = \frac{1}{2\pi i}\oint \frac{\gamma_o \bar{\varphi}'}{\bar{\omega}'} \sum_{k=0}^{\infty} \frac{\zeta^k}{\sigma^{k+1}}\, d\sigma = \frac{\gamma_o}{2\pi}\int_0^{2\pi} \sum_{l=3}^{\infty}\sum_{k=0}^{\infty} \frac{\mu_l}{\sigma^l}\frac{\zeta^k}{\sigma^k}\, d\theta = 0$$

If the values of the coefficients in the equations (65) are known, then we can solve these equations so that $c, \gamma_1, \gamma_2, \gamma_3, \ldots$ are determined up to a factor of proportion. This means that the mapping $\omega(\zeta)$ has been found up to a similarity-transformation. In case of central symmetry clearly the coefficients $\gamma_o, \gamma_2, \gamma_4, \ldots$; $a_1, a_3, a_5, \ldots$ and $\lambda_{-1}, \lambda_1, \lambda_3, \ldots$ vanish. In case of symmetry with respect to the x-axis in the z-plane the coefficients $\gamma_o, \gamma_1, \gamma_2, \ldots$ are real. Generally however we need another equation to obtain the value of γ_o. Let us again consider the equation (47):

$$G(\sigma) = 2\left[\frac{\omega}{\bar{\omega}'} \cdot \overline{\varphi'(\sigma)} + \overline{\psi(\sigma)}\right]$$

or

$$\sum_{-\infty}^{\infty} d_n\sigma^n = \frac{2\left[\frac{c}{\sigma} + \gamma_o + \gamma_1\sigma + \ldots\right]\left[\bar{a}_1 + \frac{2\bar{a}_2}{\sigma} + \frac{3\bar{a}_3}{\sigma^2} + \ldots\right]}{\left[-c\sigma^2 + \bar{\gamma}_1 + \frac{2\bar{\gamma}_2}{\sigma} + \frac{3\bar{\gamma}_3}{\sigma^2} + \ldots\right]} + 2\sum_{o}^{\infty}\frac{\bar{b}_n}{\sigma^n}$$

Thus

$$\left[-c\sigma^2 + \bar{\gamma}_1 + \frac{2\bar{\gamma}_2}{\sigma} + \frac{3\bar{\gamma}_3}{\sigma^2} + \ldots\right]\sum_{-\infty}^{\infty} d_n\sigma^n = 2\left[\frac{c}{\sigma} + \gamma_o + \gamma_1\sigma + \ldots\right]\left[\bar{a}_1 + \frac{2\bar{a}_2}{\sigma} + \frac{3\bar{a}_3}{\sigma^2} + \ldots\right]$$
$$+ 2\left[-c\sigma^2 + \bar{\gamma}_1 + \frac{2\bar{\gamma}_2}{\sigma} + \frac{3\bar{\gamma}_3}{\sigma^2} + \ldots\right]\sum_{o}^{\infty}\frac{\bar{b}_n}{\sigma^n}.$$

As regards the left member we put

$$\left[-c\sigma^2 + \bar{\gamma}_1 + \frac{2\bar{\gamma}_2}{\sigma} + \frac{3\bar{\gamma}_3}{\sigma^2} + \ldots\right]\sum_{-\infty}^{\infty} d_n\sigma^n = \sum_{-\infty}^{\infty} p_n\sigma^n$$

so that

$$p_n = -c\, d_{n-2} + \sum_{k=1}^{\infty} k\bar{\gamma}_k\, d_{n+k-1} \tag{56}$$

Thus p_k is the coefficient of σ^k in both members. It follows therefore from the right member that

$$p_k = -2\, ck\bar{a}_{-k} + 2 \sum_{r=1}^{\infty} r\, \bar{a}_r\, \gamma_{r+k-1} + 2\left[-\bar{b}_{-k+2}\, c + \sum_{r=1}^{\infty} r\, \bar{b}_{-k-r+1}\, \bar{\gamma}_r\right] \tag{57}$$

Therefore, approximating the function ψ by the first $(m+1)$ terms of its expansion (42), we obtain for $k = 0, -1, \ldots, -(m-1)$ a set of m equations in which $\gamma_0, b_2, b_3, \ldots, b_m$ are considered as the unknowns, the coefficients b_0 and b_1 being

$$b_0 = -\tfrac{1}{2}(p - q) + it \tag{58}$$

$$b_1 = -\kappa\bar{a}_1 \tag{59}$$

Thus in order to determine γ_0 it is necessary to calculate te function ψ.

Finally, as regards the determination of the factor of proportion it is observed that hitherto we did not utilize the condition (44):

$$c_0 = p + q \tag{60}$$

Thus the mapping-function $\psi(\zeta)$ must be such that this condition is satisfied.

Before occupying ourselves with the numerical treatment of the relations obtained up to now we will consider an example: Let the boundary B be a circle with radius R to which a normal loading f is applied. Because of the central symmetry we have that

$$a_1 = b_1 = 0$$

The boundary-condition (44) must be satisfied:

$$c_0 = p + q \tag{61}$$

Let the plasticity-condition be

$$\rho = k$$

where k is a constant. At the boundary B we have the stresses

$$\sigma_1 = 2k + f$$
$$\sigma_2 = f$$

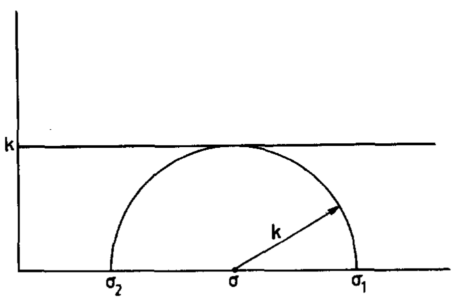

Fig. 4.

Application of the equations (32) and (33), the characteristics being logarithmic spirals $r=ae^{\pm\vartheta}$ (where $\vartheta=\varphi-\pi/2$), yields the following expressions for the stresses:

$$\begin{aligned}
\sigma_x &= \sigma + \rho\cos 2\varphi = 2k\log\frac{r}{R} + k + f - k\cos 2\vartheta \\
\sigma_y &= \sigma - \rho\cos 2\varphi = 2k\log\frac{r}{R} + k + f + k\cos 2\vartheta \\
\tau_{xy} &= \rho\sin 2\varphi = \qquad\qquad - k\sin 2\vartheta
\end{aligned}$$

from which we find that for the plastic region:

$$\sigma_x + \sigma_y = 2k\log\frac{r^2}{R^2} + 2k + 2f = 2k\log\frac{z\bar{z}}{R^2} + 2k + 2f$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2k\cos 2\vartheta + 2ik\sin 2\vartheta = 2ke^{2i\vartheta} = 2k\frac{z}{\bar{z}} \tag{62}$$

Let us consider the conformal mapping

$$z = \frac{c}{\zeta} + \gamma_0 + \gamma_1\zeta + \gamma_2\zeta^2 + \gamma_3\zeta^3 + \dots$$

Because of the central symmetry the coefficients $\gamma_0, \gamma_2, \gamma_4, \dots$ vanish. Neglecting terms of order higher than 2n we obtain the following expression for $\sigma_y - \sigma_x - 2i\tau_{xy}$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2k\,\frac{\frac{c}{\zeta} + \gamma_1\zeta + \gamma_3\zeta^3 + \dots + \gamma_{2n-1}\zeta^{2n-1}}{\frac{c}{\bar{\zeta}} + \bar{\gamma}_1\bar{\zeta} + \bar{\gamma}_3\bar{\zeta}^3 + \dots + \bar{\gamma}_{2n-1}\bar{\zeta}^{2n-1}} =$$

$$2k\bar{\zeta}\left[\frac{c}{\zeta} + \gamma_1\zeta + \gamma_3\zeta^3 + \dots + \gamma_{2n-1}\zeta^{2n-1}\right]\left[\bar{x}_0 + \bar{x}_2\bar{\zeta}^2 + \bar{x}_4\bar{\zeta}^4 + \dots\right]$$

where

$$\frac{1}{c + \gamma_1\zeta^2 + \gamma_3\zeta^4 + \gamma_5\zeta^6 + \dots + \gamma_{2n-1}\zeta^{2n}} = x_0 + x_2\zeta^2 + x_4\zeta^4 + \dots$$

From

$$1 = \left[c + \gamma_1\zeta^2 + \gamma_3\zeta^4 + \dots + \gamma_{2n-1}\zeta^{2n}\right]\left[x_0 + x_2\zeta^2 + x_4\zeta^4 + \dots\right]$$

we obtain the relations

$$\left.\begin{aligned}
&x_0 c = 1 \\
&x_0\gamma_1 + x_2 c = 0 \\
&x_0\gamma_3 + x_2\gamma_1 + x_4 c = 0 \\
&\text{-------} \\
&\text{-------} \\
&x_0\gamma_k + x_2\gamma_{k-2} + \dots + x_{k-4}\gamma_3 + \gamma_{k-2}\gamma_1 + x_k c = 0 \\
&\text{-------} \\
&\text{-------}
\end{aligned}\right\} \tag{63}$$

Thus the Fourier-series $G(\sigma)$ is

$$\frac{2k}{\sigma}\left[\frac{c}{\sigma} + \gamma_1\sigma + \gamma_3\sigma^3 + \dots + \gamma_{2n-1}\sigma^{2n-1}\right]\left[\bar{x}_0 + \frac{\bar{x}_2}{\sigma^2} + \frac{\bar{x}_4}{\sigma^4} + \dots\right]$$

and consequently we have from the relations (49), assuming that t=0:

$$\left.\begin{array}{l} 2\lambda_k = d_k = 0, \qquad k \geqslant 2n-1 \\ 2\lambda_{2n-2} = d_{2n-2} = 2k\,\bar{x}_o\gamma_{2n-1} \\ 2\lambda_{2n-4} = d_{2n-4} = 2k\,(\bar{x}_o\gamma_{2n-3} + \bar{x}_2\gamma_{2n-1}) \\ 2\lambda_{2n-6} = d_{2n-6} = 2k\,(\bar{x}_o\gamma_{2n-5} + \bar{x}_2\gamma_{2n-3} + \bar{x}_4\gamma_{2n-1}) \\ \text{etc.,} \\ \text{-------} \\ \text{-------} \\ 2\lambda_o \quad = d_o + p - q = 2k(\bar{x}_o\gamma_1 + \bar{x}_2\gamma_3 + \bar{x}_4\gamma_5 + \ldots + \bar{x}_{2n-2}\gamma_{2n-1}) + p - q \end{array}\right\} \quad (64)$$

Let us consider the equations (55):

$$\left.\begin{array}{lrcl} k=2: & 2\bar{a}_2\gamma_3 + 4\bar{a}_4\gamma_5 + \ldots + (2n-2)\bar{a}_{2n-2}\gamma_{2n-1} & = & -\lambda_o c + \lambda_2\bar{\gamma}_1 + \ldots + (2n-1)\lambda_{2n}\bar{\gamma}_{2n-1} \\ k=4: & 2\bar{a}_2\gamma_5 + \ldots + (2n-4)\bar{a}_{2n-4}\gamma_{2n-1} & = & -\lambda_2 c + \lambda_4\bar{\gamma}_1 + \ldots + (2n-1)\lambda_{2n+2}\bar{\gamma}_{2n-1} \\ \text{etc.,} & \text{-------} & & \\ & \text{-------} & & \\ k=2n-4: & 2\bar{a}_2\gamma_{2n-3} + 4\bar{a}_4\gamma_{2n-1} & = & -\lambda_{2n-6}c + \lambda_{2n-4}\bar{\gamma}_1 + \ldots + (2n-1)\lambda_{4n-6}\bar{\gamma}_{2n-1} \\ k=2n-2: & 2\bar{a}_2\gamma_{2n-1} & = & -\lambda_{2n-4}c + \lambda_{2n-2}\bar{\gamma}_1 + \ldots + (2n-1)\lambda_{4n-4}\bar{\gamma}_{2n-1} \\ k=2n: & 0 & = & -\lambda_{2n-2}c + \lambda_{2n}\bar{\gamma}_1 + \ldots + (2n-1)\lambda_{4n-2}\bar{\gamma}_{2n-1} \end{array}\right\} \quad (65)$$

Now $\lambda_k=0$ for $k \geqslant 2n-1$. From the last equation in (65) it therefore follows that $\lambda_{2n-2}=0$. From the second relation in (64) we then find that $\gamma_{2n-1}=0$. Consequently we obtain from the $(n-1)^{th}$ equation in (65) that $\lambda_{2n-4}=0$ so that from the third relation in (64) it is found that $\gamma_{2n-3}=0$ etc. In this way we find that

$$\lambda_o = \lambda_2 = \lambda_4 = ---- = 0$$

and

$$\gamma_3 = \gamma_5 = \gamma_7 = ---- = 0$$

so that

$$\omega(\zeta) = \frac{c}{\zeta} + \gamma_1\zeta$$

From $\lambda_o=0$ we obtain

$$2k\bar{x}_o\gamma_1 + p - q = 0$$

Thus

$$\gamma_1 = \frac{q-p}{2k}\,c = \beta c$$

where

$$\beta = \frac{q-p}{2k}$$

In order to determine the factor c we have the condition (61). From

$$\sigma_x + \sigma_y = 2k \log c^2(1+\beta\zeta)(1+\beta\bar{\zeta}) + 2k + 2f - 2k \log R^2$$

we find for $|\zeta| = 1$:

$$\sigma_x + \sigma_y = 2k \log c^2(1+\beta\sigma)(1+\frac{\beta}{\sigma}) + 2k + 2f - 2k \log R^2$$

Therefore

$$c_o = 2k \log \frac{c^2}{R^2} + 2k + 2f$$

Consequently

$$p + q = 2k \log \frac{c^2}{R^2} + 2k + 2f$$

from which we find for the constant c:

$$c = Re^{\frac{1}{2k}\left[\frac{p+q}{2} - f - k\right]}$$

Thus the mapping is given by

$$z = c\left(\frac{1}{\zeta} + \beta\zeta\right) = Re^{\frac{1}{2k}\left[\frac{p+q}{2} - f - k\right]}\left\{\frac{1}{\zeta} + \frac{q-p}{2k}\zeta\right\}$$

which function represents an ellipse with semi-axes $c(1+\beta)$ and $c(1-\beta)$.

5. *The numerical method*

To solve the elasto-plastic problem using the relations which have been derived in the preceding section a relaxation-method can be applied. To some given set of variables $c, \gamma_o, \gamma_1, \gamma_2, \ldots, \gamma_n$ corresponds a transformation-function $\omega_1(\zeta)$ that maps a curve C_1 onto the unit-circle in the complex ζ-plane. Let us start from the idea that the equations (34), (35) and (36) refer to the contour C_1. Application of the equations (29) and (30) yields stresses in the region bounded by C_1. Thus from a (generally numerical) Fourier-analysis on the left members of the equations (34), (35) and (36) coefficients c_k and d_k are available. Then from the relations (43) coefficients a_k, $k \geq 1$ follow and from the relations (49), (50) and (51) coefficients λ_k, $k \geq -1$ are obtained. In this way coefficients a_k and λ_k are added to the set of variables $c, \gamma_o, \gamma_1, \gamma_2, \ldots \gamma_n$.

If the equations (55) and (58) and the condition (44) are denoted by $f_i(c, \gamma_o, \gamma_1, \gamma_2, \ldots \gamma_n) = 0$ then the positive semi-definite function defined by

$$S = \sum_i |f_i|$$

equals zero for the solution of the problem. Therefore one can find this solution by minimizing the function S; starting from an initial point on the surface S one has to change in some way the variables $c, \gamma_o, \gamma_1, \gamma_2, \ldots, \gamma_n$ such that S decreases. Having thus found a new point the process is repeated. In order to study some more details let us consider an example: Let the boundary B be a circle with radius a to which a normal loading f is applied. Because of the central symmetry $a_1 = b_1 = 0$. Let the plasticity-condition be

$$\rho = (\sigma + \sigma_o) \sin \beta$$

At the boundary B we have the stresses

$$\sigma_1 = f(1 + 2\lambda) + 2k$$

$$\sigma_2 = f$$

where

$$k = \lambda\sigma_0$$

and

$$\lambda = \frac{\sin\beta}{1 - \sin\beta}$$

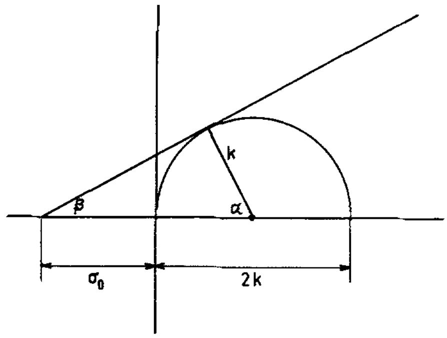

Fig. 5.

Application of the equations (32) and (33), the characteristics being logarithmic spirals $A\exp\left[\pm\,\theta\,\mathrm{tg}\,\frac{\alpha}{2}\right]$ (where $\theta = \varphi - \pi/2$) yields the following expressions for σ and ρ:

$$\sigma = (1+\lambda)\,f\,\left(\frac{r}{a}\right)^{2\lambda} + k\,\frac{\left(\frac{r}{a}\right)^{2\lambda} - 1}{\lambda} + k\,\left(\frac{r}{a}\right)^{2\lambda}$$

$$\rho = \lambda f\,\left(\frac{r}{a}\right)^{2\lambda} + k\,\left(\frac{r}{a}\right)^{2\lambda}$$

Thus we obtain from

$$\begin{aligned} \sigma_x &= \sigma + \rho\cos 2\varphi = \sigma - \rho\cos 2\vartheta \\ \sigma_y &= \sigma - \rho\cos 2\varphi = \sigma + \rho\cos 2\vartheta \\ \tau_{xy} &= \rho\sin 2\varphi = -\rho\sin 2\vartheta \end{aligned}$$

that for the plastic region

$$\sigma_x + \sigma_y = 2\,\frac{\left[\lambda f(1+\lambda) + k(1+\lambda)\right]\left(\frac{r}{a}\right)^{2\lambda} - k}{\lambda}$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2\left[\lambda f + k\right]\left(\frac{r}{a}\right)^{2\lambda} e^{2i\vartheta}$$

Let us consider the case that $a=1$, $k=1$, $\beta=\pi/6$, so that $\lambda=1$. We then have the following simple expressions for the plastic region:

$$\sigma_x + \sigma_y = 2(2\,z\bar{z} - 1) \tag{66}$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2z^2 \tag{67}$$

Therefore, approximating the mapping function (37) by

$$z = \frac{c}{\zeta} + \gamma_1\zeta + \gamma_3\zeta^3 + \gamma_5\zeta^5 + \gamma_7\zeta^7 + \gamma_9\zeta^9$$

(the coefficients $c, \gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9$ beeing real because of the symmetry with respect to the x-axis in the z-plane) we find that

$$\sigma_x + \sigma_y = 2\left[2\left(\frac{c}{\sigma} + \gamma_1\sigma + \gamma_3\sigma^3 + \gamma_5\sigma^5 + \gamma_7\sigma^7 + \gamma_9\sigma^9\right)\left(c\sigma + \frac{\gamma_1}{\sigma} + \frac{\gamma_3}{\sigma^3} + \frac{\gamma_5}{\sigma^5} + \frac{\gamma_7}{\sigma^7} + \frac{\gamma_9}{\sigma^9}\right) - 1\right]$$

Consequently we obtain from the relations (43):

$$\begin{aligned} 2a_2 &= c_2 = 4\,(c\gamma_1 + \gamma_1\gamma_3 + \gamma_3\gamma_5 + \gamma_5\gamma_7 + \gamma_7\gamma_9) \\ 2a_4 &= c_4 = 4\,(c\gamma_3 + \gamma_1\gamma_5 + \gamma_3\gamma_7 + \gamma_5\gamma_9) \\ 2a_6 &= c_6 = 4\,(c\gamma_5 + \gamma_1\gamma_7 + \gamma_3\gamma_9) \\ 2a_8 &= c_8 = 4\,(c\gamma_7 + \gamma_1\gamma_9) \end{aligned} \tag{68}$$

and from the boundary-condition (44):

$$4\,(c^2 + \gamma_1^2 + \gamma_3^2 + \gamma_5^2 + \gamma_7^2 + \gamma_9^2) - 2 = p + q \tag{69}$$

From the expression:

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2\,\left(\frac{c}{\sigma} + \gamma_1\sigma + \gamma_3\sigma^3 + \gamma_5\sigma^5 + \gamma_7\sigma^7 + \gamma_9\sigma^9\right)^2.$$

we obtain as a consequence of the relations (49) and (50), assuming that $t=0$:

$$\begin{aligned} 2\lambda_0 &= d_0 + p - q = 4c\gamma_1 + p - q \\ 2\lambda_2 &= d_2 = 2\gamma_1^2 + 4c\gamma_3 \\ 2\lambda_4 &= d_4 = 4c\gamma_5 + 4\gamma_1\gamma_3 \\ 2\lambda_6 &= d_6 = 2\gamma_3^2 + 4\gamma_1\gamma_5 + 4c\gamma_7 \\ 2\lambda_8 &= d_8 = 4\gamma_3\gamma_5 + 4\gamma_1\gamma_7 + 4c\gamma_9 \\ 2\lambda_{10} &= d_{10} = 2\gamma_5^2 + 4\gamma_3\gamma_7 + 4\gamma_1\gamma_9 \\ 2\lambda_{12} &= d_{12} = 4\gamma_5\gamma_7 + 4\gamma_3\gamma_9 \\ 2\lambda_{14} &= d_{14} = 2\gamma_7^2 + 4\gamma_5\gamma_9 \\ 2\lambda_{16} &= d_{16} = 4\gamma_7\gamma_9 \\ 2\lambda_{18} &= d_{18} = 2\gamma_9^2 \end{aligned} \tag{70}$$

Finally we have from (55) the equations:

$$\left.\begin{aligned} k=2:\ f_1 &= -\lambda_0 c + \lambda_2\gamma_1 + (3\lambda_4 - 2a_2)\gamma_3 + (5\lambda_6 - 4a_4)\gamma_5 + (7\lambda_8 - 6a_6)\gamma_7 + (9\lambda_{10} - 8a_8)\gamma_9 = 0 \\ k=4:\ f_2 &= -\lambda_2 c + \lambda_4\gamma_1 + 3\lambda_6\gamma_3 + (5\lambda_8 - 2a_2)\gamma_5 + (7\lambda_{10} - 4a_4)\gamma_7 + (9\lambda_{12} - 6a_6)\gamma_9 = 0 \\ k=6:\ f_3 &= -\lambda_4 c + \lambda_6\gamma_1 + 3\lambda_8\gamma_3 + 5\lambda_{10}\gamma_5 + (7\lambda_{12} - 2a_2)\gamma_7 + (9\lambda_{14} - 4a_4)\gamma_9 = 0 \\ k=8:\ f_4 &= -\lambda_6 c + \lambda_8\gamma_1 + 3\lambda_{10}\gamma_3 + 5\lambda_{12}\gamma_5 + 7\lambda_{14}\gamma_7 + (9\lambda_{16} - 2a_2)\gamma_9 = 0 \\ k=10:\ f_5 &= -\lambda_8 c + \lambda_{10}\gamma_1 + 3\lambda_{12}\gamma_3 + 5\lambda_{14}\gamma_5 + 7\lambda_{16}\gamma_7 + 9\lambda_{18}\gamma_9 = 0 \end{aligned}\right\} \tag{71}$$

The unknowns $c, \gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9$ must satisfy the condition:

$$2(c^2 + \gamma_1^2 + \gamma_3^2 + \gamma_5^2 + \gamma_7^2 + \gamma_9^2) - 1 = \frac{p+q}{2} \tag{72}$$

It is observed that in this example a numerical Fourier-analysis is not necessary; from (68) and (70) the coefficients in the equations (71) are available from given values of $c, \gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9$.

If $p=q$ then the equations (71) are solved by $\gamma_1=\gamma_3=\gamma_5=\gamma_7=\gamma_9=0$ so that from (72) we obtain for the constant c:

$$c = \sqrt{\tfrac{1}{2} + \frac{p+q}{4}} \tag{73}$$

Thus is case $p=q$ the curve separating the elastic from the plastic region is a circle with radius $\left\{\frac{1}{2} + \frac{p+q}{4}\right\}^{\frac{1}{2}}$.

To find for different values of p and q the solution of the problem relaxation is applied. A simple method is to minimize the function

$$S(c, \gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9) = \sum_{i=1}^{5} \left| f_i(c, \gamma_1, \gamma_3, \gamma_5, \gamma_7, \gamma_9)\right|$$

by changing alternately the variables in the direction of their axes such that S decreases. After such a modification of the variables a correction must be applied because of the condition (72). However such a correction can disturb the improvement of the function S. This can be avoided by putting

$$f_6 = 2(c^2 + \gamma_1^2 + \gamma_3^2 + \gamma_5^2 + \gamma_7^2 + \gamma_9^2) - 1 - \frac{p+q}{2}$$

and minimizing the function

$$S = \sum_{i=1}^{6} |f_i|$$

Furthermore this simple method generally breaks down because of ridges that occur on the surface S. Then it is essential to start a new stage, i.e. to change the directions in which the variables are modified. This is done in Rosenbrock's process, which process was used to solve the equations (71) with the condition (72). As regards the starting point it is observed that if $p=q$ the solution is known from (73). Therefore it is recommendable to vary the quantities p and q starting from $p=q$; the start of the next calculation is then the result of the preceding one. The following results were obtained:

c	γ_1	γ_3	γ_5	γ_7	γ_9	p	q	accuracy	nr. of stages
1,414	0	0	0	0	0	3	3		0
1,3911	-0,044	-0,0006	0	0	0	3	2,75	0,01	3
1,3912	-0,0450	-0,0007	0	0	0	3	2,75	0,001	4
1,3657	-0,092	-0,0035	-0,0004	0	0	3	2,5	0,01	3
1,3386	-0,1415	-0,0053	-0,0006	0	0	3	2,25	0,01	10
1,3082	-0,1961	-0,0182	-0,0029	-0,0005	-0,0001	3	2	0,01	7
1,2733	-0,2555	-0,0274	-0,0059	-0,0019	-0,0006	3	1,75	0,01	6
1,414	0	0	0	0	0	3	3		0
1,403	-0,1809	-0,0113	-0,0013	-0,0002		3,5	2,5	0,01	3
1,385	-0,2835	-0,0328	-0,0076	-0,0025	-0,0012	3,75	2,25	0,01	7

c	γ_1	γ_3	γ_5	γ_7	γ_9	p	q	accuracy	nr. of stages
1,732	0	0	0	0	0	5	5		0
1,7260	-0,1458	-0,0056	-0,0005	-0,0001	0	5,5	4,5	0,01	4
1,7043	-0,3034	-0,0292	-0,0058	-0,0014	-0,0003	6	4	0,01	4
1,7050	-0,3034	-0,0290	-0,0056	-0,0014	-0,0004	6	4	0,00002	10

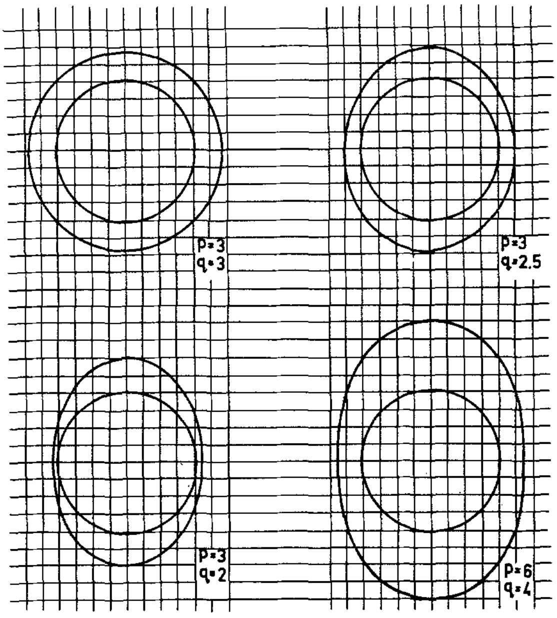

Fig. 6.

For high values of the stresses p and q the approximation of $\omega(\zeta)$ appeared to be insufficient. It is then necessary that some more coefficients in the approximation are taken into account. Starting from the mapping-function

$$z = \frac{c}{\zeta} + \gamma_1\zeta + \gamma_3\zeta^3 + \gamma_5\zeta^5 + \gamma_7\zeta^7 + \gamma_9\zeta^9 + \gamma_{11}\zeta^{11} + \gamma_{13}\zeta^{13} + \gamma_{15}\zeta^{15}$$

one obtains the required equations in a similar way as in the preceding example. By minimizing the function

$$S = \sum_{i=1}^{9} |f_i|$$

where eight functions $f_1, f_2, \ldots, f_8$ are obtained from the equations (55) and f_9 is constituted by the condition (44) the following results appear

c	γ_1	γ_3	γ_5	γ_7	γ_9	γ_{11}	γ_{13}	γ_{15}	p	q
2,35	0	0	0	0	0	0	0	0	10	10
2,289	-0,109	-0,003	-0,0001	0	0	0	0	0	10	9
2,226	-0,225	-0,012	-0,0013	-0,0002	0	0	0	0	10	8
2,150	-0,360	-0,032	-0,006	-0,0013	-0,0004	-0,0001	-0,0001	0	10	7
2,053	-0,527	-0,081	-0,026	-0,011	-0,0052	-0,0027	-0,0014	-0,0007	10	6

c	γ_1	γ_3	γ_5	γ_7	γ_9	γ_{11}	γ_{13}	γ_{15}	p	q
3,606	0	0	0	0	0	0	0	0	25	25
3,495	-0,216	-0,006	-0,0004	0	0	0	0	0	25	22
3,405	-0,371	-0,021	-0,0022	-0,0003	0	0	0	0	25	20
3,306	-0,545	-0,048	-0,0088	-0,0023	-0,0006	-0,0002	0	0	25	18
3,190	-0,752	-0,102	-0,028	-0,01	-0,0041	-0,0018	-0,0008	-0,0004	25	16
3,1053	-0,904	-0,175	-0,0757	-0,0427	-0,0272	-0,0183	-0,0119	-0,0067	25	15

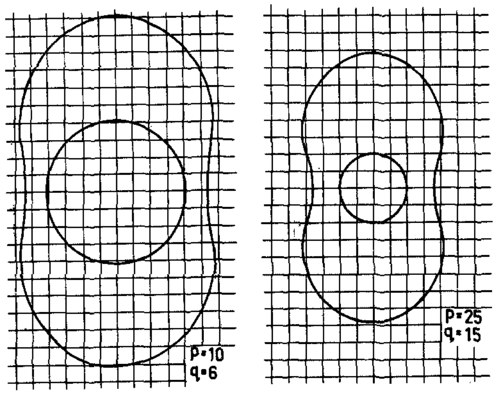

Fig. 7.

To improve accuracy one could try to apply successive approximation. However generally the matrix of coefficients in the equations (55) is ill-conditioned.

Finally it is observed that a relaxation-method as pointed out above can be generally applied to obtain the solution of the elasto-plastic problem.

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